

The Longitudinal Analysis of Poverty Conceptualized as a Fuzzy State

Analisi Longitudinale della Povertà Vista come Stato Sfocato

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Riassunto: Uno dei limiti principali delle misure convenzionali di povertà riguarda la rigida classificazione poveri/non poveri su cui esse si basano. Per trattare la povertà come una questione di grado, e dunque come uno stato sfocato, occorre affrontare due problemi: i) modellizzare il grado di povertà di ciascuna unità osservata e ii) definire nel modo più opportuno le operazioni logiche di unione, intersezione e complemento con cui manipolare i risultanti insiemi sfocati. Il presente articolo tratta in modo specifico di questa seconda questione, con particolare attenzione al contesto longitudinale. Risolto questo problema, si possono costruire varie misure sfocate di povertà che rappresentano altrettante generalizzazioni delle corrispondenti misure convenzionali.

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1. Introduction

Conventional analyses of poverty often have two main limitations: (i) they are unidimensional, considering only income poverty, and (ii) they need to dichotomize the population into the poor and the non-poor by means of a poverty line. While multidimensionality is being increasingly incorporated into poverty research, little attention has been devoted to the effect of the rigid poor/non-poor dichotomy on the results.

When poverty is viewed as a matter of degree (as distinct from the conventional poor/non-poor dichotomy), that is as a fuzzy state, two additional aspects are introduced into the analysis. These concern the choice of (i) membership functions i.e. quantitative specification of individuals' or households' degrees of poverty; (ii) rules for the manipulation of the resulting fuzzy sets, rules defining their complements, intersections, union and averaging.

We address the second of the above questions in this paper. Specifically, for longitudinal analysis of poverty using the fuzzy set approach, we need longitudinal joint membership functions covering more than one time period, which have to be constructed on the basis of the series of cross-sectional membership functions over those time periods.

For an early application of the ideas of fuzzy sets to the longitudinal study of poverty, see Cheli (1995); some further development of those ideas can be found in Betti, Cheli and Cambini (2004).

In this paper, we take a fresh look at the methodology and develop some basic rules concerning fuzzy set operations, as relevant for the longitudinal analysis of poverty.

It is useful to begin by a brief clarification of the concept of treating poverty (or more generally, various forms of deprivation) as a matter of degree replacing the conventional classification of the population into a simple dichotomy. In the conventional approach the population is dichotomized as $\{0,1\}$: those with income below a certain threshold are deemed to be poor (i.e. are all assigned a constant propensity=1); others with income at or above that threshold are deemed to be non-poor (i.e. are all assigned a constant propensity=0). The fuzzy approach is a generalization of it. Here in principle all individuals in a population are subject to poverty, but to varying degrees. We say that each individual has a certain propensity to be poor, the population covering the whole range $[0,1]$.

Our motivation in conceptualising poverty as a matter of degree in the continuum $[0,1]$ is essentially practical and empirical. The objective is to construct measures more adequately reflective of disparities in income in particular at the lower end of the distribution. The measures need to be sensitive to those disparities, at the same time be robust against small changes in the level of income. Such robustness is particularly important in the analysis of persistence over time and other longitudinal aspects of poverty.¹

At the same time it is essential that, to be useful in practice, the new ‘fuzzy’ measures are consistent and comparable, to the maximum extent possible, with the well-established – albeit less satisfactory – conventional measures of the status and level of poverty. This requires that the basic rules for the construction and manipulation of the fuzzy measures parallel, or at least be closely linked to, the corresponding rules for the conventional measures. The former have to be meaningful generalisations which reproduce the latter when the degree of poverty $[0,1]$ is reduced to a simple ‘yes-no’ or $\{0,1\}$ dichotomy.

It is useful to begin by describing the conventional analysis in terms which facilitate the formalisation of its meaningful and consistent fuzzy generalisation. The conventional conceptualisation has the following aspects. The rest of the paper is concerned with fuzzy generalisation of each of these.

(1) The poverty status of an individual i at time t can be described in terms of a dichotomous variable $\mu_{t,i}^{(1)} = \{0,1\}$, where the value 1 identifies a poor and 0 a non-poor person.

(2) The average $\bar{\mu}_t^{(1)}$ over i gives the proportion in poverty, i.e. the cross-sectional poverty rate or ‘head count ratio’ at t .

(3) Equally, we can describe an individual’s status with complement of (1): $1 - \mu_{t,i}^{(1)} = \mu_{t,i}^{(0)} = \{0,1\}$, where now value 1 identifies a non-poor and 0 a poor individual.

(4) Obviously, this complementary at the micro-level implies the same at the aggregate (average) level: $\bar{\mu}_t^{(0)} = 1 - \bar{\mu}_t^{(1)}$. In fact, individuals in a given set are partitioned into two

¹ Formally the same requirements apply in the study of overlap between different – monetary and non-monetary – dimensions of deprivation though we do not pursue this issue further in this paper.

non-overlapping and exhaustive subsets: those with $\mu_{t,i}^{(1)} = 1$ (or $\mu_{t,i}^{(0)} = 0$) and those with $\mu_{t,i}^{(0)} = 1$ (or $\mu_{t,i}^{(1)} = 0$), and their proportions add up to unity: $\bar{\mu}_t^{(1)} + \bar{\mu}_t^{(0)} = 1$.

(5) The status of an individual in terms of persistent poverty, for instance over two years 1 and 2, may be described with $\bigcap_i^{(1,1)} = \mu_{1,i}^{(1)} \cap \mu_{2,i}^{(1)} = \{0,1\}$, which clearly equals 1 when both the components μ_1 and μ_2 equal 1, i.e. the person is poor at both times.

Similarly $\bigcup_i^{(1,1)} = \mu_{1,i}^{(1)} \cup \mu_{2,i}^{(1)}$ indicates the status of being poor at any of the two times, as it equals 1 when either of the two components equals 1.

(6) The averages of these quantities give the corresponding rates in the population. For example, $\bar{\bigcap}_i^{(1,1)}$ is the proportion in persistent poverty and $\bar{\bigcup}_i^{(1,1)}$ the proportion in poverty at least one of the two years.

(7) For a given individual, all possible intersections involving various states and their complements add up to 1, in the manner of (4). For example, considering two time periods, $\bigcap_i^{(1,1)} + \bigcap_i^{(1,0)} + \bigcap_i^{(0,1)} + \bigcap_i^{(0,0)} = 1$.

In fact, for given i , only one of the terms above equals 1; the others are 0. As in (4), individuals in a given set are partitioned (in this example) into four non-overlapping and exhaustive subsets, and the proportions in the various categories add up to 1. The above concepts can be generalised without alteration to any series of cross-sections $t = (1, \dots, T)$.

2. Individual propensities to poverty in the cross-sectional context

Now we consider step-by-step generalization of the above concept in terms of the individuals' propensities to be poor, viewed as membership of a fuzzy set: $\mu_{t,i} \in [0,1]$.

Replacing (1) in the previous section, the basic idea is as follows.

Given a set X of elements $x \in X$, any fuzzy subset A of X is defined as: $A = \{x, \mu_A(x)\}$, where $\mu_A(x): X \rightarrow [0,1]$ is called the membership function (m.f.) in the fuzzy subset A . The value $\mu_A(x)$ indicates the degree of membership of x in A . Thus $\mu_A(x) = 0$ means that x does not belong at all to A , whereas $\mu_A(x) = 1$ means that x belongs to A completely. When on the other hand $0 < \mu_A(x) < 1$, then x partially belongs to A and its degree of membership of A increases in proportion to the proximity of $\mu_A(x)$ to 1.

We begin from the concept that, for a series of cross-sections $(1, \dots, T)$, each person's propensity to be in poverty (i.e. the person's membership function in the set 'poor') is given as $\mu_{t,i}$, ($t = 1, \dots, T$), $\mu_{t,i} \in [0,1]$.

In line with (2), it is useful to view the average of these memberships as a cross-sectional 'poverty rate' for the population at any given time t . And as in (3), we can also define, where necessary, the complement of the above, $\mu'_{t,i} = 1 - \mu_{t,i}$, being the individual's membership of the complementary set. In this definition, the original set and its complement are taken to form 'fuzzy partitions' of the universal set X . On the basis of these membership functions (m.f.) the individuals are no longer partitioned into non-overlapping or exhaustive (crisp) subsets – in fact any individual belongs to each set, but only to a degree.

However, the important implication is that – as a consequence of the individual memberships being constrained by the requirement for the sets to be fuzzy partitions – the averages of these memberships add up to 1 as in (4): $\bar{\mu}_t + \bar{\mu}'_t = 1$.

Obviously this propensity is essential for a meaningful interpretation of these averages as complementary proportions or rates – in this example the rates of poverty and non-poverty, respectively, which must add up to 1 to be meaningful.

This completes the conceptualisation from a cross-sectional point of view. The first practical task is to define the functional form of the m.f. $\mu_{t,i}$. This is a substantive issue.

We propose in Section 6 a particular form where $\mu_{t,i}$ is a function of the individual's position in the income distribution as well as the share of total income possessed by those less poor than the individual concerned. Substantive considerations also dictate that the functional form be more sensitive to the bottom end of the income distribution. In fact, it is highly desirable that its average be closely linked to the conventional poverty rate so as to facilitate comparisons of the results of the two methodologies. In fact we propose to define the m.f. such that, in our present terminology, $\bar{\mu}_t = \bar{\mu}_t^{(1)}$, or more generally averaged such as over a time interval, $\bar{\bar{\mu}} = \bar{\bar{\mu}}^{(1)}$. Superscript (1) here indicates the conventional poverty rate, for instance.

3. Fuzzy-set operations in the longitudinal context: basic rules

For analysis of poverty over time, we need joint membership functions (j.m.f.'s) covering more than one time period. Simple examples are the intersection $\mu_1 \cap \mu_2$ giving the j.m.f. of the set 'poor at the both times 1 and 2', and the union $\mu_1 \cup \mu_2$ giving the j.m.f. of the set 'poor at either of the two times'. Similarly $\mu_1 \cap \mu'_2$ gives the propensity of being poor at time 1 but of escaping from poverty by time 2, and $\mu'_1 \cap \mu_2$ gives the propensity of being non-poor at time 1, but falling into poverty by time 2.

Just as the mean of individual values such as μ_t of the m.f.'s can be seen as the (cross-sectional) poverty rate at time t, the mean of a j.m.f., of for instance $\mu_1 \cap \mu_2$ gives the rate of persistent poverty over the two years.

While fuzzy set operations are a generalisation of the corresponding 'crisp' set operations, there is more than one way in which the fuzzy set operations can be formulated, each representing an equally valid generalisation of the corresponding crisp set operation. The choice among alternative formulations has to be made primarily on substantive grounds: some options are more appropriate (meaningful, useful, illuminating, convenient) than others, depending on the context and objectives of the application. While the rules of fuzzy set operations cannot be discussed fully in this paper, we need to clarify their application specifically for the study of poverty and deprivation.

There are four types of fuzzy set operations on membership functions which are relevant to our application to longitudinal poverty analysis: (i) fuzzy intersection, (ii) fuzzy union, (iii) fuzzy complement, and (iv) aggregation (or averaging) over fuzzy sets. There are three commonly-used groups of rules – termed Standard, Algebraic and Bounded (Klir and Yuan, 1995) – specifying fuzzy intersection and union. Such rules

are ‘permissible’ in the sense that they satisfy certain essential requirements such as reducing to the crisp set operations with dichotomous variables, satisfying the required boundary conditions, being monotonic and commutative, etc.

Let us consider these operations applied to a pair of fuzzy sets, and use simple and general notation as follows. Quantities a and b refer to the membership function of a given individual on the pair of fuzzy sets, relative to two time periods 1 and 2.² Also, a and b may represent similar states (e.g. poor, poor), or dissimilar states (poor, non-poor), or even states differing to various degrees. We can also define their complements, the m.f.'s of the corresponding ‘non-poor’ sets, as $a' = 1 - a$, $b' = 1 - b$. In longitudinal analysis, four intersection sets can be formed from these, with their joint membership functions under different set operations defined in Table 1.

Table 1: *Application of different types of fuzzy intersections over two times periods*

Fuzzy intersection	Standard operator	Algebraic operator	Bounded operator	Betti-Verma Composite operator		
				Type	Intersection	Union
$a \cap b$	$\min(a,b)$	$a*b$	$\max(0,a+b-1)$	Standard	$\min(a,b)$	$\max(a,b)$
$a' \cap b'$	$\min(1-a, 1-b) = 1 - \max(a,b)$	$(1-a)*(1-b)$	$\max(0, 1-a-b)$	Standard	$1 - \max(a,b)$	$1 - \min(a,b)$
$a \cap b'$	$\min(a, 1-b)$	$a*(1-b)$	$\max(0, a-b)$	Bounded	$\max(0, a-b)$	$1 - \max(0, b-a)$
$a' \cap b$	$\min(1-a, b)$	$(1-a)*b$	$\max(0, b-a)$	Bounded	$\max(0, b-a)$	$1 - \max(0, a-b)$

The results of these operations need to satisfy certain marginal constraints of the type in (4) and (7) for instance.

The specific ones corresponding to the intersections in Table 1 are laid out in Table 2.

Table 2: *2x2 fuzzy intersections and marginal constraints*

	b	b'	marginal
a	$a \cap b$	$a \cap b'$	a
a'	$a' \cap b$	$a' \cap b'$	$a' = 1 - a$
marginal	b	$b' = 1 - b$	1

The commonly used Standard operator does not meet these constraints. In fact this operator provides the largest intersection and the smallest union among all the permitted forms. If the Standard operation were applied to all the four intersections, the sum of membership functions of an individual can be verified to equal $1 + 2(s_1 - \max(0, \delta))$, where $s_1 = \min(a,b)$, and $\delta = (a+b-1)$, i.e., to equal $(1+2s_1)$ for $\delta \leq 0$, and $1+2(1-s_2)$ for $\delta > 0$, where $s_2 = \max(a,b)$.

Now it can be easily seen that among the commonly used operators, the Algebraic form, applied to all the four intersections, is the only one which meets this condition. But despite this numerical consistency, we do not regard the Algebraic form to give results which, for our particular application, would be generally acceptable on intuitive or substantive grounds. In fact, if we take the liberty of viewing the fuzzy propensities as

² Equally to two different dimensions of deprivation at the same time: the formal rules are identical in the two situations.

probabilities, then the algebraic product rule $a \cap b$ implies zero correlation between the two forms of deprivation, which is clearly at variance with the high positive correlation we expect in the real situation for similar states. The rule therefore seems to provide an unrealistically low estimate for the resulting membership function for the intersection of two similar states. The Standard rule, giving higher overlaps (intersections) are more realistic for (a,b) representing similar states.

By contrast, in relation to dissimilar states (a', b) and (a, b') (lack of an overlap between deprivations in two dimensions), the Algebraic rule tend to give unrealistically high estimates for the resulting membership function for the intersection, and hence this is true also the Standard rule. The reasoning similar to the above applies: in real situations, we expect large negative correlations (hence reduced intersections) between dissimilar states in the two dimensions of deprivation. In fact, it can be easily seen by considering some particular numerical values for (a', b) or (a, b') that Bounded rule, for instance, gives more realistic results for dissimilar states.

Given the preceding considerations, the specification of the fuzzy intersection $a \cap b$ that appears to be the most reasonable for our particular application and that satisfies the above mentioned marginal constraints is of a 'composite' type as follows:

For sets representing similar states - such as the presence (or absence) of poverty at both times - the Standard operation (which provides a larger intersection than the Algebraic operation) is used.

For sets representing dissimilar states - such as the presence of poverty at one time but its absence at the other time - we use the Bounded operation (which provides a smaller intersection than the Algebraic operation). Betti and Verma (2004) term this operator as Composite. Note that the satisfaction of the marginal constraints implies that the intersection operators are distributive: $a \cap b + a \cap b' = a \cap (b + b') = a$. This is obviously true of the Algebraic operators. It is also true of the Betti-Verma Composite operator since $a \cap b + a \cap b' = \min(a, b) + \max(0, a - b) = a$. Any other type of operators, including the Standard and Bounded ones, are not distributive in this sense. Table 3 shows results of the application of the Composite operator.

Note that the propensity to be ever in poverty equals $\max(a, b)$, which can be viewed as any of the three entirely equivalent forms: as the complement of 1; as the sum of the membership functions in 2, 3 and 4; as the union $a_i \cup b_i$.

Table 3: *Longitudinal measures of interest over two time periods for individual i*

	Measure	Membership function	Description
1	Never in poverty	$a'_i \cap b'_i = 1 - \max(a_i, b_i)$	Poverty at <i>neither</i> of the two years
2	Persistent in poverty	$a_i \cap b_i = \min(a_i, b_i)$	Poverty at <i>both</i> of the years
3	Exiting from poverty	$a_i \cap b'_i = \max(0, a_i - b_i)$	Poverty at time 1, but non-poverty at time 2
4	Entering into poverty	$a'_i \cap b_i = \max(0, b_i - a_i)$	Non-poverty at time 1, but poverty at time 2
5	Ever in poverty	$a_i \cup b_i = \max(a_i, b_i)$	Poverty at <i>at least one</i> of the two years

4. Longitudinal poverty over a sequence of time periods

We begin below by formulating a general rule for defining the intersection of any arbitrary sequence of cross-sectional propensities or degrees of membership. Subsequently we will explore its characteristics and applications. The rule distinguishes between individuals' degrees of membership in fuzzy sets of two opposite types such as 'poor' and 'non-poor', and is a generalization of the Composite operator defined earlier. Consider any sequence of cross-sectional propensities. It can always be expressed in the form $(\dots, \mu_{t_1}, \dots) (\dots, \mu_{t_2}, \dots)$, where t_1 indicates T_1 elements of the same type in one group, and t_2 indicates T_2 elements of the opposite type in the other group; clearly $T_1 + T_2 = T$. The generalized Composite rule for defining the intersection (joint membership function) of the sequence of T cross-sectional propensities (membership functions) is as follows:

- (i) Sort the elements into two groups by type, for instance all T_1 elements of one type followed by all T_2 elements of the other type.
- (ii) Construct the intersection for each group involving elements of the same type using the Standard operator.
- (iii) Finally, construct the intersection of the two results of the above operation using the Bounded operator.

Note that the operator is independent of the temporal order of the T elements, we may view the application of this rule as being 'without memory'. More precisely perhaps, we may designate it as a procedure 'without chronology': the outcome depends on the whole 'history' (i.e., the specified type of cross-sectional sets in the time sequence $t=1$ to T , and the associated membership functions); but it does not depend on the actual chronology, the temporal sequence, of those cross-sections.

As an illustration of the application of the generalised composite operator, to consider the longitudinal situation involving 3 time-periods. The $2^3 = 8$ intersections are shown in Table 4.

Again we use simplified notation, with (a, b, c) as the cross-sectional propensities to poverty at year 1, 2 and 3, respectively, and (a', b', c') as their complements.

Table 4 shows only a subset of marginal constraints, each concerning a pair of complementing sets involving c and c' , such as the pair $(a \cap b \cap c, a \cap b \cap c')$.

The marginals are simply the 4 intersections determined by applying the Composite operator to the first 2 time periods (Tables 2 and 3). Note that they sum to 1 for any individual as required by the overall constraint.

Table 4: *Membership functions for the 8 intersections sets for 3 time-periods*

		c	c'	marginal
a	b	(1) $\min(a, b, c)$	(2) $\max(0, \min(a, b) - c)$	(1)+(2) = $\min(a, b)$
	b'	(3) $\max[0, \min(a, c) - b]$	(4) $\max(0, a - \max(b, c))$	(3)+(4) = $\max(0, a - b)$
a'	b	(5) $\max(0, \min(b, c) - a)$	(6) $\max[0, b - \max(a, c)]$	(5)+(6) = $\max(0, b - a)$
	b'	(7) $\max(0, c - \max(a, b))$	(8) $1 - \max(a, b, c)$	(7)+(8) = $1 - \max(a, b)$

The 3-period intersections in the internal cells of Table 4 have been obtained simply by applying the above rule. It can be easily verified that the expressions in the cells of a row add up to the row total. For instance:

$$a \cap b \cap c + a \cap b \cap c' = \min(a, b, c) + \max(0, \min(a, b) - c) = \min(a, b) = a \cap b.$$

By interchanging a, b and c among themselves, or by permuting their order additional marginal constraints (in fact 27 in all) can be identified: these are all similarly satisfied in this example. It can be easily shown that the generalized composite rule ensures that all marginal constraints of the type being discussed are satisfied in the general case (Betti *et al.*, 2006).

5. Longitudinal measures of poverty

In this section examples are provided of some commonly-used longitudinal measures constructed using the fuzzy conceptualization of poverty and the generalized composite operator. Our starting point are the individual propensities to poverty $\mu_{t,i}$ at a series of cross-sections $t = 1$ to T .

Analysis of the persistence of poverty over time requires the specification of j.m.f.'s of the type $I_{T,i} = \mu_{1,i} \cap \mu_{2,i} \dots \dots \cap \mu_{T,i}$ and $U_{T,i} = \mu_{1,i} \cup \mu_{2,i} \dots \dots \cup \mu_{T,i}$, where the first expression is the intersection of a series of T cross-sectional m.f.'s for any individual unit, and the second expression is their union.

Since all sets $\mu_1, \dots, \mu_T$ are of the same type (all being propensities to 'poverty' rather than to 'non-poverty'), their intersection is given simply by the Standard operator:

$$I_T = \min(\mu_1, \mu_2, \dots, \mu_t, \dots, \mu_T) \quad (1)$$

$$U_T = \max(\mu_1, \mu_2, \dots, \mu_t, \dots, \mu_T). \quad (2)$$

Clearly these expressions are commutative and associative. I_T represents the individual's propensity to be poor at all T periods. U_T is the propensity to be poor at *at least one* of the T periods; the propensity to be non-poor over all T periods is its complement $U'_T = 1 - U_T$. The same result is obtained by considering intersection of non-poor sets:

$$\bar{I}_T = \min(\mu'_1, \mu'_2, \dots, \mu'_t, \dots, \mu'_T) = 1 - \max(\mu_1, \mu_2, \dots, \mu_T) = 1 - U_T.$$

The propensity to be poor in exactly t out of T years is the sum of j.m.f.'s over all sequences with t cross-sectional sets of the type 'poor' and the remaining $(T-t)$ of the type 'non-poor'. For any particular sequence of this type, rearrange the sets such that the first t terms are of the type 'poor'. With the generalized composite rule, the j.m.f. for the particular sequence is: $\mu_i = \max[0, \min(\mu_{1,i}, \mu_{2,i}, \dots, \mu_{t,i}) - \max(\mu_{t+1,i}, \mu_{t+2,i}, \dots, \mu_{T,i})]$

which is non-zero only for one sequence in which the first group contains the t largest m.f.'s. With $[t]$ denoting the ordered sequence of decreasing μ values, the required j.m.f. becomes:

$$\text{Poor (exactly } t \text{ out of } T \text{ years): } \mu[t] - \mu[t+1],$$

and by simple addition:

$$\text{Poor (at least } t \text{ out of } T \text{ years): } \mu[t], \text{ the } t^{\text{th}} \text{ largest value.}$$

Any time poverty

We may define the anytime poverty rate of a population as the average of the individual degrees of membership in the set 'poor for at least one year' = $\mu[1]$.

Continuous poverty

The continuous or ever poverty rate for a population is given by the average of the individual joint memberships for the set intersection, estimated for example as the membership function of the set ‘poor for all the T years’ = $\mu[T]$.

These are particular cases of the propensity to be poor for at least t out of T years = $\mu[t]$, the t^{th} largest value.

Persistent poverty

We may define persistent poverty as the propensity to be poor over at least a majority of the T years. The required propensity to persistent poverty is the $[\text{int}(T/2)+1]^{\text{th}}$ largest value in the sequence $(\mu_1, \mu_2, \dots, \mu_T)$. For instance, for a T= 4 or 5 year period, ‘persistent’ would refer to poverty for at least 3 years; for T =6 or 7, it would refer to poverty for at least 4 years, etc.

6. Application to Italy

We provide numerical illustration of the various types of longitudinal measures which can be constructed using the above formulations.

The data used for all illustrations are from Italian European Community Household Panel Survey 1994-2001. The tables in this section also show some measures of substantive interest for Italy and her Micro-regions. We compare fuzzy measures against the corresponding conventional measures.

For the computation presented, each unit (person) is classified as poor or non-poor in relation to a poverty line defined as 60% of the national median income. (These computations are performed separately for each wave or each pair of consecutive waves of the survey, and then the results are averaged over those.)

For each survey wave, the membership function (μ_i) for the set ‘poor’ has been specified for each individual i (indexed according to increasing size of equivalized income y_i) following the ‘Integrated Fuzzy and Relative’ (IFR) approach developed in Betti, Cheli, Lemmi and Verma (2005) which combines the two approaches of Cheli and Lemmi (1995) and Betti and Verma (1999)³. The details of this approach will not be described in this paper, except to note that the fuzzy membership function for (or the propensity to) income poverty is formulated so as to take into account both the share of individuals less poor than the person concerned and the share of the total equivalized income received by all individuals less poor than the person concerned:

$$\mu_i = (1 - F_i)^{\alpha-1} [1 - L(F_i)] = \left(\frac{\sum_{\gamma} w_{\gamma} | y_{\gamma} > y_i}{\sum_{\gamma} w_{\gamma} | y_{\gamma} > y_1} \right)^{\alpha-1} \left(\frac{\sum_{\gamma} w_{\gamma} y_{\gamma} | y_{\gamma} > y_i}{\sum_{\gamma} w_{\gamma} y_{\gamma} | y_{\gamma} > y_1} \right), \quad (3)$$

³ Although we are convinced that an adequate analysis of poverty should be carried out according to a multidimensional approach that include a variety of living conditions indicators of the non-monetary type, for the sake of this research it will be enough to confine our attention to the analysis of income poverty alone.

where F_i is the distribution function of the equivalized income, $L(F_i)$ is the corresponding Lorenz ordinate, and parameter α is chosen so that the mean of the μ_i equals to conventional head count ratio H . The measure as defined above is expressible in terms of the generalized Gini measures (the standard Gini coefficient corresponds to G_α with $\alpha=1$), which weights the distance $(F-L(F))$ between the line of perfect equality and the Lorenz curve by a function of the individual's position in the income distribution, giving more weight to its poorer end. Is defined (in the continuous case) as:

$$\bar{\mu} = \frac{\alpha + G_{\alpha}}{\alpha(\alpha + 1)} = H. \quad (5)$$

Table 5: Conventional and fuzzy cross-sectional measures of the income poverty rate

ECHP wave	w1	w2	w3	w4	w5	w6	w7	w8	8 ECHP waves	
Fuzzy monetary (FM) poverty rate									FM	FM/H
Italy	19.4	19.4	19.3	19.3	19.3	19.2	19.2	19.2	19.3	1.00
North-West	11.5	11.0	10.4	10.3	10.9	9.1	9.1	9.5	10.2	1.05
North-East	13.0	11.4	9.9	10.0	8.9	9.2	8.7	8.7	10.0	1.26
Center	15.4	14.7	15.2	16.1	16.4	15.8	16.4	17.2	15.9	1.14
South	29.3	30.2	30.2	30.8	30.0	31.8	30.8	30.2	30.4	0.92
Islands	34.2	32.9	35.2	32.9	33.9	35.0	36.3	35.9	34.5	0.89

Using a balanced panel over 8 waves of the Italian ECHP, the above procedures have been applied to estimate the following measures.⁴

- Rate of continuous poverty, i.e. poverty over all of the 8 years covered in the panel.
- Rate of persistent poverty, i.e. poverty over at least 5 of the 8 years.
- Rate of any-time poverty, i.e. experience of poverty over 1 or more of the 8 years.

As before we have constructed these measures for Italy and her Macro-regions using Italian ECHP survey. Numerical results, also comparing these measures using the conventional (poor/non-poor) approach with those using the fuzzy approach, are presented in Table 6.

Noteworthy from a methodological view point is the difference in the performance of the conventional and the fuzzy approaches, especially concerning the estimated incidence of continuous poverty. It appears that movements in and out of poverty tend to be somewhat over-estimated (and hence the persistent or continuous poverty rates under-estimated) with the conventional approach, presumably because it gives too much weight even to small movements across the poverty line.

Table 6: *Longitudinal income poverty rates*

Macro-Region	Anytime	Persistent	Continuous	Anytime	Persistent	Continuous
North-West	26.5	2.8	0.4	25.7	5.8	1.7
North-East	26.5	3.3	0.5	24.4	6.3	2.2
Center	41.1	9.1	1.3	36.1	12.6	4.0
South	60.0	25.5	5.7	52.0	24.8	9.4
Islands	67.1	31.8	9.5	56.9	29.1	12.7
Italy	42.6	13.2	3.0	37.8	14.7	5.5

7. Concluding remarks

When poverty is viewed as a matter of degree in contrast to the conventional poor/non-poor dichotomy, that is, as a fuzzy state, two additional aspects are introduced into the analysis.

- (i) The choice of membership functions i.e. quantitative specification of individuals' or households' degrees of poverty and deprivation.
- (ii) And the choice of rules for the manipulation of the resulting fuzzy sets, rules defining their complements, intersections, union and averaging. Specifically, for longitudinal analysis of poverty using the fuzzy set approach, we need joint membership functions covering more than one time period, which have to be constructed on the basis of the series of cross-sectional membership functions over those time periods.

We have aimed to address in this paper the second of the above questions.

We have proposed a general rule for the construction of fuzzy set intersections, that is for the construction of a longitudinal poverty measures from a sequence of cross-sectional measures under fuzzy conceptualisation. This general rule is meant to be applicable to any sequence of 'poor' and 'non-poor' set, and it satisfies all the marginal constraints.

⁴ A balanced panel means that only individuals present in all the 8 waves in the sample are retained in the analysis.

This is an important contribution since there are reasons to believe that, hitherto, the rules of fuzzy set operations in the context of multi-dimensional and longitudinal poverty analysis have not been well or widely understood.

On the basis of the results obtained, various fuzzy poverty measures over time can be constructed as consistent generalisations of the corresponding conventional (dichotomous) measures.

In conclusion we want to emphasize again the potential practical usefulness of the approach in particular in yielding more robust and stable measures of change and persistence over time.

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